JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **191**, 250–265 (1995)

Transformations of Linear Hamiltonian Difference Systems and Some of Their Applications

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Submitted by Hal L. Smith

Received June 9, 1993; revised April 26, 1994

1. INTRODUCTION

Consider the linear Hamiltonian difference system (LHS)

$$\begin{aligned} \Delta y_n &= A_n y_{n+1} + B_n z_n, \\ \Delta z_n &= C_n y_{n+1} - A_n^T z_n, \end{aligned} \tag{1.1}$$

where y_n, z_n are d-dimensional ($d \in \mathbb{N}$) sequences, A_n, B_n, C_n are sequences of real-valued $d \times d$ matrices, $B_n^T = B_n$, $C_n^T = C_n$ (the superscript "T" stands for the transpose of the matrix indicated), Δ is the usual forward difference operator, and $n \in [M-1, \infty)$, $M \in \mathbb{N}$.

Recently, considerable effort has been made to find analogies between qualitative properties of solutions of (1.1) and those of the continuous LHS

$$\begin{aligned} y' &= A(t)y + B(t)z, \\ z' &= C(t)y - A^T(t)z, \end{aligned} \tag{1.2}$$

cf. [2, 3, 7–9, 12, 13]; particularly, in [2] an extensive bibliography concerning the problem may be found (concerning the basic properties of continuous LHS (1.2), the reader is referred to monographs [4, 14]). It was shown that certain discrete quadratic functional and Riccati matrix difference equations play essentially the same role as their continuous counterparts in the continuous case.

The aim of this paper is to show that for the investigation of difference

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LHS we may use a transformation method which is similar to that given for continuous LHS, e.g., in [1, Section VI]. We also use this method in order to study singular discrete quadratic functionals corresponding to (1.1).

In [10] it is conjectured that the positivity of the quadratic functional corresponding to (1.1) with B_n only nonnegative definite implies the existence of a symmetric solution of the associated Riccati equation. In case of the positive definite matrix B_n this result is proved and applied to the investigation of oscillation properties of (1.1) in [7–9].

Our transformation method which requires the matrix B_n to be only nonnegative definite may turn out to be useful in solving the conjecture of [10], which also covers an important case when (1.1) corresponds to the $2d$-order scalar difference equation

$$\sum_{k=0}^{d} (-1)^k \Delta^k (p_n^k \Delta^k y_{n+d-k}) = 0,$$

where $\Delta^k = \Delta(\Delta^{k-1})$, $p_n^d > 0$, which can be written in the form (1.1) with the matrix $B_n = \text{diag}\,\{0, \ldots, 0, 1/p_n^d\}$.

The paper is organized as follows. In the next section we give some basic properties of solutions of (1.1) and we also recall the results concerning the continuous LHS (1.2) whose discrete analogies are proved in Sections 3 and 4. In Section 3 the general form of the transformation of (1.1) is introduced which transforms this system into the system of the same form and it is shown that when B_n is positive definite and $(I - A_n)$ is nonsingular, I being the identity $d \times d$ matrix, (1.1) may be transformed into a LHS corresponding the to vector second order difference equation $\Delta^2 y_n + P_n y_{n+1} = 0$, where P_n is a symmetric $d \times d$ matrix. In the last section the singular discrete quadratic functional corresponding to (1.1) is investigated. Particularly, the discrete analogy of the so-called singularity condition is introduced, which together with disconjugacy of (1.1) implies nonnegativity of the investigated functional. However, in this section we need the assumption of nonsingularity of $(I - A_n)$ and positive definiteness of B_n; the open problem is whether the results of this kind hold also for LHS and quadratic functionals with B_n only nonnegative definite (as it is the case for continuous functionals).

2. Preliminary Results

Simultaneously with system (1.1) consider its matrix version

$$\begin{aligned} \Delta Y_n &= A_n Y_{n+1} + B_n Z_n, \\ \Delta Z_n &= C_n Y_{n+1} - A_n^T Z_n, \end{aligned} \tag{2.1}$$

where Y_n, Z_n are $d \times d$ matrices. Any $2d \times d$ solution $X_n = \begin{pmatrix} Y_n \\ Z_n \end{pmatrix}$ of (2.1) we will write in the form $X_n = \{Y_n, Z_n\}$.

Let $X_n = \{Y_n, Z_n\}$ be a solution of (2.1). Then $\Delta(X_n^T \mathcal{J} X_n) = \Delta(Y_n^T Z_n - Z_n^T Y_n) = 0$, where

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.2}$$

i.e., $Y_n^T Z_n - Z_n^T Y_n \equiv K$ – a constant $d \times d$ matrix. If $K = 0$, the solution $\{Y_n, Z_n\}$ is said to be *prepared.* In the continuous case, the alternative terminology is *isotropic* solution [4] or *self-conjoined* solution [14]. The term prepared solution is due to Hartman [11].

The following basic properties of solutions of (1.1) and (1.2) which we need in the sequel are proved in the papers [7–10]. Let $\{Y_n, Z_n\}$ be a prepared solution of (2.1) such that Y_n is nonsingular. Then $W_n = Z_n Y_n^{-1}$ is a symmetric solution of the Riccati matrix difference equation

$$\begin{aligned} \Delta W_n + (I - A_n)^T W_n (I + B_n W_n)^{-1} B_n W_n (I - A_n) \\ - C_n + W_n A_n + A_n^T W_n + A_n^T W_n A_n = 0. \end{aligned} \tag{2.3}$$

Throughout the paper, (A) and (B) mean the following hypotheses:

(A) The matrix $(I - A_n)$ is nonsingular.

(B) The matrix B_n is positive definite.

Up to the end of this section we suppose that these hypotheses are satisfied.

System (1.1) is said to be *disconjugate* on an interval $[M - 1, N + 1]$ if for any nontrivial solution of this system there exists at most one $k \in [M - 1, N]$ such that

$$Y_k^T B_k^{-1} (I - A_k) Y_{k+1} \le 0;$$

in the opposite case (1.1) is said to be *conjugate* on $[M - 1, N + 1]$. System (1.1) is said to be *eventually disconjugate* if there exists $M \in \mathbb{N}$ such that (1.1) is disconjugate on $[M - 1, N + 1]$ for every $N > M$. A prepared solution $\{Y_n, Z_n\}$ of (2.1) is said to be *recessive* (*dominant*) at ∞ if

$$Y_n^T B_n^{-1} (I - A_n) Y_{n+1} > 0 \tag{2.4}$$

for large n and

$$\lim_{n\to\infty}\left[\sum_{k=M}^{n} Y_{k+1}^{-1}(I-A_k)^{-1}B_k Y_k^{T-1}\right]=0$$

$$\left(\lim_{n\to\infty}\left[\sum_{k=M}^{n} Y_{k+1}^{-1}(I-A_k)^{-1}B_k Y_k^{T-1}\right]=K-\text{a nonsingular matrix}\right).$$

Similar to the continuous case, the discrete quadratic functional

$$q(y_n, z_n; M-1, N)=\sum_{k=M-1}^{N}[z_n^T B_n z_n + y_{n+1}^T C_n y_{n+1}],$$

where

$$\{y_n, z_n\}\in\Omega(M-1, N)$$
$$:=\{y_n, z_n\in\mathbb{R}^d, y_{M-1}=0=y_{N+1}, \Delta y_n = A_n y_{n+1}+B_n z_n, n\in[M-1, N]\}$$

plays an important role as the following propositions show.

PROPOSITION 2.1 [9, Theorem 1]. *The following are equivalent:*

(i) *System* (1.1) *is disconjugate on* $[M-1, N+1]$.

(ii) *The functional* $q(y_n, z_n; M-1, N)$ *is positive definite on* $\Omega(M-1, N)$.

(iii) *There exists a symmetric solution of* (2.3) *such that* $B_n^{-1}+W_n>0$ *for* $n\in[M-1, N]$.

(iv) *There exists a prepared solution* $\{Y_n, Z_n\}$ *of* (2.1) *such that* (2.4) *holds for* $n\in[M-1, N]$.

PROPOSITION 2.2 [9, Theorem 2]. *Let* (1.1) *be eventually disconjugate. Then:*

(i) *Every prepared solution* $\{Y_n, Z_n\}$ *of* (2.1) *for which rank* $\{Y_n, Z_n\}=d$ *satisfies* (2.4) *for n sufficiently large.*

(ii) *There exists a solution* $\{Y_n, Z_n\}$ *of* (2.1) *which is recessive (dominant) at* ∞.

At the end of this section we recall (for the sake of comparison) the continuous counterparts of Theorems 3.1 and 4.1 given below.

PROPOSITION 2.3 [1, Theorem 6.3]. *Let* $\mathcal{R}(t)\in C^1$ *be a* $\mathcal{J}$*-unitary* $2d\times 2d$ *matrix (i.e.,* $\mathcal{R}^T(t)\mathcal{J}\,\mathcal{R}(t)=\mathcal{J}$, $\mathcal{J}$ *is given by* (2.2)) *consisting of* $d\times d$ *matrices* H, K, M, N:

$$\mathcal{R}(t)=\begin{pmatrix}H(t) & M(t)\\ K(t) & N(t)\end{pmatrix}.$$

The transformation $y = H(t)u + M(t)v$, $z = K(t)u + N(t)v$ *transforms* (1.2) *into the LHS*

$$\begin{aligned} u' &= \tilde{A}(t)u + \tilde{B}(t)v, \\ v' &= \tilde{C}(t)u - \tilde{A}^T(t)v, \end{aligned}$$

where

$$\begin{aligned} \tilde{A}(t) &= N^T(t)[-H'(t) + A(t)H(t) + B(t)K(t)] \\ &\quad - M^T(t)[-K'(t) + C(t)H(t) - A^T(t)K(t)], \\ \tilde{B}(t) &= N^T(t)[-M'(t) + A(t)M(t) + B(t)N(t)] \\ &\quad + M^T(t)[-N'(t) + C(t)M(t) - A^T(t)N(t)], \\ \tilde{C}(t) &= -K^T(t)[-H'(t) + A(t)H(t) + B(t)K(t)] \\ &\quad + H^T(t)[-K'(t) + C(t)H(t) - A^T(t)K(t)]. \end{aligned}$$

PROPOSITION 2.4 [5, Theorem 2]. *Suppose that* $B(t) \geq 0$, (1.2) *is identically normal on* $[\alpha, \infty)$ (*i.e., if* $y(t) \equiv 0$ *on a nondegenerate subinterval of* $[a, \infty)$ *for some solution* $\{y(t), z(t)\}$ *of* (1.2) *then* $\{y(t), z(t)\} \equiv \{0, 0\}$ *for* $t \in [a, \infty)$). *Then*

$$J(y, z, a, \infty) = \liminf_{b \to \infty} \int_a^b [z^T(t)B(t)z(t) + y^T(t)C(t)y(t)]\, dt \geq 0$$

over all

$$\begin{aligned} \{y, z\} \in D &:= \{y(t), z(t) \in R^d, y \in AC[a, \infty), z \in L^2_{\mathrm{loc}}[a, \infty), \\ &\quad y(a) = 0 = \lim_{t \to \infty} y(t) \text{ and } y' = A(t)y + B(t)z\} \end{aligned}$$

if and only if

(i) (1.2) *is disconjugate on* $[a, \infty)$ (*i.e., if for some solution* $\{y(t), z(t)\}$ *of* (1.2) *and some distinct* $t_1, t_2 \in [a, \infty)$, $y(t_1) = y(t_2) = 0$ *then* $y(t) \equiv 0$ *for* $t \in [a, \infty)$), *and*

(ii) *the singularity condition*

$$\liminf_{t \to \infty} y^T(t)Z(t)Y^{-1}(t)y(t) \geq 0$$

holds for every $\{y, z\} \in D$ *for which* $J(y, z, a, \infty) < \infty$, *where* $\{Y(t), Z(t)\}$ *is the solution of the matrix LHS* (1.2) *for which* $Y(a) = 0$, $Z(a) = I$.

3. Transformations of Difference LHS

THEOREM 3.1. *Let $\mathcal{R}_n$ be a $2d \times 2d$ $\mathcal{J}$-unitary matrix consisting of $d \times d$ matrices*

$$\mathcal{R}_n = \begin{pmatrix} H_n & M_n \\ K_n & N_n \end{pmatrix} \tag{3.1}$$

such that the matrix

$$\begin{pmatrix} H_n + B_n K_n & (I - A_n) M_{n+1} \\ (I - A_n^T) K_n & N_{n+1} - C_n M_{n+1} \end{pmatrix} \tag{3.2}$$

is nonsingular and denote

$$\begin{pmatrix} H_n + B_n K_n & (I - A_n) M_{n+1} \\ (I - A_n^T) K_n & N_{n+1} - C_n M_{n+1} \end{pmatrix}^{-1} =: \begin{pmatrix} D_n & F_n \\ E_n & G_n \end{pmatrix}. \tag{3.3}$$

The transformation

$$y_n = H_n u_n + M_n v_n, \qquad z_n = K_n u_n + N_n v_n \tag{3.4}$$

transforms (1.1) *into the system*

$$\begin{aligned} \Delta u_n &= \tilde{A}_n u_{n+1} + \tilde{B}_n v_n, \\ \Delta v_n &= \tilde{C}_n u_{n+1} - \tilde{A}_n^T v_n, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \tilde{A}_n &= D_n(-\Delta H_n + A_n H_{n+1} + B_n K_n) + F_n(-\Delta K_n + C_n H_{n+1} - A_n^T K_n), \\ \tilde{B}_n &= D_n(-\Delta M_n + A_n M_{n+1} + B_n N_n) + F_n(-\Delta N_n + C_n M_{n+1} - A_n^T N_n), \\ \tilde{C}_n &= E_n(-\Delta H_n + A_n H_{n+1} + B_n K_n) + G_n(-\Delta K_n + C_n H_{n+1} - A_n^T K_n), \end{aligned} \tag{3.6}$$

whereby the matrices $\tilde{B}_n$, $\tilde{C}_n$ are symmetric, i.e., (3.5) *is again a LHS.*

Proof. To simplify the following computations, denote

$$x_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}, \tilde{x}_n = \begin{pmatrix} y_{n+1} \\ z_n \end{pmatrix}, w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tilde{w}_n = \begin{pmatrix} u_{n+1} \\ v_n \end{pmatrix},$$

$$\mathcal{H}_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n^T \end{pmatrix}, \mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (1.1) and (3.4) become $\Delta x_n = \mathcal{H}_n \tilde{x}_n$, $x_n = \mathcal{R}_n w_n$, respectively. We have

$$\begin{aligned}\Delta x_n &= \Delta\mathcal{R}_n w_n + \mathcal{R}_{n+1}\Delta w_n = \Delta\mathcal{R}_n \tilde{w}_n - \Delta\mathcal{R}_n \mathcal{J}\Delta w_n + \mathcal{R}_{n+1}\Delta w_n \\ &= \Delta\mathcal{R}_n \tilde{w}_n + (\mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{J})\Delta w_n\end{aligned} \tag{3.7}$$

and using this relation

$$\begin{aligned}\Delta x_n &= \mathcal{H}_n \tilde{x}_n \\ &= \mathcal{H}_n(x_n + \mathcal{J}\,\Delta x_n) = \mathcal{H}_n[\mathcal{R}_n w_n + \mathcal{J}\,\Delta\mathcal{R}_n\tilde{w}_n + \mathcal{J}(\mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{J})\,\Delta w_n] \\ &= \mathcal{H}_n[\mathcal{R}_n(\tilde{w}_n - \mathcal{J}\,\Delta w_n) + \mathcal{J}\,\Delta\mathcal{R}_n\tilde{w}_n + \mathcal{J}(\mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{J})\,\Delta w_n] \\ &= \mathcal{H}_n[(\mathcal{R}_n + \mathcal{J}\,\Delta\mathcal{R}_n)\tilde{w}_n + (-\mathcal{R}_n\mathcal{J} + \mathcal{J}\mathcal{R}_{n+1} - \mathcal{J}\,\Delta\mathcal{R}_n\mathcal{J})\,\Delta w_n].\end{aligned}$$

Combining the last computation with (3.7), we have

$$\begin{aligned}&[\mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{J} + \mathcal{H}_n(\mathcal{R}_n\mathcal{J} - \mathcal{J}\mathcal{R}_{n+1} + \mathcal{J}\,\Delta\mathcal{R}_n\mathcal{J})]\,\Delta w_n \\ &\qquad = [-\Delta\mathcal{R}_n + \mathcal{H}_n(\mathcal{R}_n + \mathcal{J}\,\Delta\mathcal{R}_n)]\tilde{w}_n.\end{aligned} \tag{3.8}$$

Denote

$$\begin{aligned}\mathcal{L}_n &= \mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{J} + \mathcal{H}_n(\mathcal{R}_n\mathcal{J} - \mathcal{J}\mathcal{R}_n + \mathcal{J}\,\Delta\mathcal{R}_n\mathcal{J}), \\ \mathcal{K}_n &= -\Delta\mathcal{R}_n + \mathcal{H}_n(\mathcal{R}_n + \mathcal{J}\,\Delta\mathcal{R}_n).\end{aligned}$$

Substituting for $\mathcal{R}_n$ from (3.1) we see that $\mathcal{L}_n$ equals the matrix (3.2) which is assumed to be nonsingular, hence the system which results from (1.1) upon the transformation (3.4) may be written in the form

$$\Delta w_n = \mathcal{L}_n^{-1}\mathcal{K}_n\tilde{w}_n.$$

In order to verify that this is also a LHS of the form (1.1) we need to show that

$$(\mathcal{L}_n^{-1}\mathcal{K}_n)^T\mathcal{J} + \mathcal{J}\mathcal{L}_n^{-1}\mathcal{K}_n = 0,$$

since concerning the system (1.1),

$$\mathcal{H}_n^T\mathcal{J} + \mathcal{J}\mathcal{H}_n = \begin{pmatrix} -C_n^T & A_n^T \\ A_n & B_n^T \end{pmatrix} + \begin{pmatrix} C_n & -A_n^T \\ -A_n & -B_n \end{pmatrix} = 0.$$

We have

$$(\mathcal{L}_n^{-1}\mathcal{K}_n)^T\mathcal{J} + \mathcal{J}\mathcal{L}_n^{-1}\mathcal{K}_n = -\mathcal{J}\mathcal{L}_n^{-1}(\mathcal{L}_n\mathcal{J}\mathcal{K}_n^T + \mathcal{K}_n\mathcal{J}\mathcal{L}_n^T)\mathcal{L}_n^{T-1}\mathcal{J}$$

and

$$\begin{aligned}
\mathcal{L}_n\mathcal{J}\mathcal{K}_n^T + \mathcal{K}_n\mathcal{J}\mathcal{L}_n^T = {} & [\mathcal{R}_{n+1} - \Delta\mathcal{R}_n\mathcal{I} + \mathcal{H}_n(\mathcal{R}_n\mathcal{I} \\
& - \mathcal{I}\mathcal{R}_{n+1} + \mathcal{I}\mathcal{R}_n\mathcal{I})]\mathcal{J}[-\Delta\mathcal{R}_n^T + (\mathcal{R}_n^T + \Delta\mathcal{R}_n^T\mathcal{J})\mathcal{H}_n^T] \\
& + [-\Delta\mathcal{R}_n + \mathcal{H}_n(\mathcal{R}_n + \mathcal{I}\,\Delta\mathcal{R}_n)]\mathcal{J}[\mathcal{R}_{n+1}^T \\
& - \mathcal{I}\,\Delta\mathcal{R}_n^T + (\mathcal{I}\mathcal{R}_n^T - \mathcal{R}_{n+1}^T\mathcal{J} + \mathcal{I}\,\Delta\mathcal{R}_n^T\mathcal{I})\mathcal{H}_n^T] \\
= {} & -\mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T - \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T + \Delta\mathcal{R}_n(\mathcal{I}\mathcal{J} \\
& + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n^T + [\mathcal{R}_{n+1}\,\mathcal{J}\mathcal{R}_n^T - \Delta\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\mathcal{R}_n^T \\
& + (-\Delta\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n^T - \Delta\mathcal{R}_n\,\mathcal{J}\mathcal{R}_{n+1}^T \\
& + \mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T)]\mathcal{H}_n + \mathcal{H}_n[-\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n^T \\
& + \mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T + \mathcal{I}(-\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n + \mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T \\
& + \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T)] + \mathcal{H}_n[\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\mathcal{R}_n^T \\
& + (\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n^T - \mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T)\mathcal{J} \\
& + \mathcal{I}(-\mathcal{R}_{n+1}\mathcal{J}\mathcal{R}_n^T + \Delta\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\mathcal{R}_n^T) \\
& + \mathcal{I}(-\mathcal{R}_{n+1}\mathcal{J}\mathcal{R}_n^T + \Delta\mathcal{R}_n(\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I})\,\Delta\mathcal{R}_n^T \\
& - \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T)\mathcal{I}]\mathcal{H}_n = -\mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T - \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T \\
& + [\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T + (-\mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T - \mathcal{R}_n\mathcal{J}\mathcal{R}_n^T)\mathcal{I}]\mathcal{H}_n^T \\
& + \mathcal{H}_n[\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T + \mathcal{I}(\mathcal{R}_n\mathcal{J}\,\Delta\mathcal{R}_n^T + \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T)] \\
& + \mathcal{H}_n[\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T - \mathcal{R}_n\mathcal{J}\mathcal{R}_n^T\mathcal{I} - \mathcal{I}\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T \\
& + \mathcal{I}(-\mathcal{R}_{n+1}\mathcal{J}\mathcal{R}_n^T + \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_n^T)\mathcal{I}]\mathcal{H}_n = 0 + \mathcal{J}\mathcal{H}_n^T \\
& + \mathcal{H}_n\mathcal{J} + \mathcal{H}_n(\mathcal{J} - \mathcal{J}\mathcal{I} - \mathcal{I}\mathcal{J})\mathcal{H}_n^T \\
= {} & -\mathcal{J}(\mathcal{H}_n^T\mathcal{J} + \mathcal{J}\mathcal{H}_n)\mathcal{J} = 0,
\end{aligned}$$

where the identities

$$\mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I} = \mathcal{J},\ \mathcal{R}_n\mathcal{J}\mathcal{R}_n^T = \mathcal{J}, \qquad \mathcal{R}_{n+1}\mathcal{J}\,\Delta\mathcal{R}_n^T + \Delta\mathcal{R}_n\mathcal{J}\mathcal{R}_{n+1}^T = 0$$

have been used.

Now, substituting for $\mathcal{R}_n$ in $\mathcal{K}_n$, we have

$$\mathcal{K}_n = \begin{pmatrix} -\Delta H_n + A_n H_{n+1} + B_n K_n & -\Delta M_n + A_n M_{n+1} + B_n N_n \\ -\Delta K_n + C_n H_{n+1} + A_n^T K_n & -\Delta N_n + C_n M_{n+1} + A_n^T N_n \end{pmatrix}$$

and hence the matrices $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ in $\mathcal{L}_n^{-1}\mathcal{K}_n$ are just of the form (3.6). ∎

If $M_n \equiv 0$ in (3.1) then transformation (3.4) preserves the oscillation behaviour of the transformed LHS and we have the following statement.

COROLLARY 3.1. *Let $M_n \equiv 0$, then $N_n = H_n^{T-1}$ and the matrix* (3.1) *is nonsingular if and only if $H_n + B_n K_n$ is nonsingular. The matrices $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ are of the form*

$$\begin{aligned} \tilde{A}_n &= I - (H_n + B_n K_n)^{-1}(I - A_n)H_{n+1} \\ \tilde{B}_n &= (H_n + B_n K_n)^{-1} B_n H_n^{T-1} \\ \tilde{C}_n &= H_{n+1}^T [-K_{n+1} + C_n H_{n+1} + (I - A_n^T)K_n(H_n + B_n K_n)^{-1}(I - A_n)H_{n+1}] \end{aligned} \tag{3.9}$$

Proof. If $M_n \equiv 0$ then one may directly verify that

$$\begin{aligned} D_n &= (H_n + B_n K_n)^{-1}, \\ E_n &= -H_{n+1}^T(I - A_n^T)K_n(H_n + B_n K_n)^{-1}, \\ F_n &= 0, \\ C_n &= H_{n+1}^T. \end{aligned}$$

$\mathcal{J}$-unitarity of $\mathcal{R}(t)$ implies $H_n^T N_n - K_n^T M_n = I$; i.e., if $M_n \equiv 0$ then $N_n = H_n^{T-1}$ and hence

$$\begin{aligned} \tilde{A}_n &= (H_n + B_n K_n)^{-1}(-\Delta H_n + A_n H_{n+1} + B_n K_n) \\ &= (H_n + B_n K_n)^{-1}[-(I - A_n)H_{n+1} + H_n + B_n K_n) \\ &= I - (H_n + B_n K_n)^{-1}(I - A_n)H_{n+1}, \\ \tilde{B}_n &= (H_n + B_n K_n)^{-1} B_n H_n^{T-1} \\ \tilde{C}_n &= -H_{n+1}^T(I - A_n^T)K_n((H_n + B_n K_n)^{-1}[-(I - A_n)H_{n+1} + H_n + B_n K_n)] \\ &\quad + H_{n+1}^T[(I - A_n^T)K_n - K_{n+1} + C_n H_{n+1}] \\ &= H_{n+1}^T(I - A_n^T)K_n(H_n + B_n K_n)^{-1} \\ &\quad (I - A_n)H_{n+1} - H_{n+1}^T(I - A_n)^T K_n + H_{n+1}^T(I - A_n^T)K_n - H_{n+1}^T K_{n+1} \\ &\quad + H_{n+1}^T C_n H_{n+1} = H_{n+1}^T[-K_{n+1} + C_n H_{n+1} \\ &\quad + (I - A_n^T)K_n(H_n + B_n K_n)^{-1}(I - A_n)H_{n+1}]. \quad ∎ \end{aligned}$$

Remark 3.1. Observe that transformation (3.4) with $M_n \equiv 0$ applied to the matrix system (2.1) transforms recessive (dominant) solutions into recessive (dominant) solutions. Indeed, if $\{U_n, V_n\}$ is a $2d \times d$ solution of the matrix LHS corresponding to (3.5) then

$$\begin{aligned}
&\sum^{N} U_{n+1}^{-1}(I - \tilde{A}_n)^{-1}\tilde{B}_n U_n^{T-1} \\
&= \sum^{N} Y_{n+1}^{-1} H_{n+1} H_{n+1}^{-1}(H_n + B_n K_n)(H_n + B_n K_n)^{-1} B_n H_n^{T-1} H_n^T Y_n^{T-1} \\
&= \sum^{N} Y_{n+1}^{-1}(I - A_n)^{-1} B_n Y_n^{T-1}.
\end{aligned}$$

Note also that hypotheses (A), (B) are satisfied for (1.1) if and only if the corresponding conditions are satisfied for (3.5).

COROLLARY 3.2. *Let* $\{Y_n, Z_n\}$ *be a prepared solution of* (2.1) *such that* Y_n *is nonsingular and assume that* (A) *holds. The transformation*

$$y_n = Y_n u_n, \qquad z_n = Z_n u_n + Y_n^{T-1} v_n \tag{3.10}$$

transforms (1.1) *into the system*

$$\begin{aligned}
&\Delta u_n = Y_{n+1}^{-1}(I - A_n)^{-1} B_n Y_n^{T-1} v_n, \\
&\Delta v_n = 0.
\end{aligned} \tag{3.11}$$

Consequently,

$$\begin{aligned}
\tilde{Y}_n &= Y_n \sum_{k=M}^{n} Y_{k+1}^{-1}(I - A_k)^{-1} B_k Y_k^{-1}, \\
\tilde{Z}_n &= Z_n \sum_{k=M}^{n} Y_{k+1}^{-1}(I - A_k)^{-1} B_k Y_k^{-1} + Y_n^{T-1}
\end{aligned}$$

is also a prepared solution of (2.1).

Proof. The preparedness of $\{Y_n, Z_n\}$ implies that the matrix

$$\mathcal{R}_n = \begin{pmatrix} Y_n & 0 \\ Z_n & Y_n^{T-1} \end{pmatrix}$$

is $\mathcal{J}$-unitary and $H_n + B_n K_n = Y_n + B_n Z_n - (-\Delta Y_n + A_n Y_{n+1} + B_n Z_n) = (I - A_n) Y_{n+1}$. Let $\{U_n, V_n\}$ be the $2d \times d$ solution of the matrix LHS corresponding to (3.11) for which $U_M = 0$, $V_M = I$. Then $V_n = I$ and

$$U_n = \sum_{k=M}^{n} Y_{k+1}^{-1}(I - A_k)^{-1} B_k Y_k^{-1},$$

i.e., $\{\tilde{Y}_n, \tilde{Z}_n\} = \{Y_n U_n, Z_n U_n + Y_n^{T-1} V_n\}$ is a solution of (2.1) whose preparedness may be verified by a direct computation. ▮

The following theorem is a discrete analogy of Theorem 2 of [6].

THEOREM 3.2. *Let* (A), (B) *hold. There exist $d \times d$ matrices H_n, K_n such that the transformation* (3.4) *with $M_n \equiv 0$ (i.e., $N_n = H_n^{T-1}$–see Corollary* 3.1.) *transforms* (1.1) *into the LHS corresponding to a second order difference equation*

$$\Delta^2 u_n + P_n u_{n+1} = 0,$$

where P_n is a symmetric $d \times d$ matrix.

Proof. Let

$$\overline{H}_M = I, \qquad \overline{H}_{n+1} = (I - A_n)^{-1} H_n, \qquad n \geq M.$$

Then $\Delta \overline{H}_n = A_n \overline{H}_{n+1}$ and the transformation

$$y_n = \overline{H}_n \tilde{u}_n, \qquad z_n = \overline{H}_n^{T-1} \tilde{v}_n$$

transforms (1.1) into the system

$$\Delta \tilde{u}_n = \overline{H}_n^{-1} B_n \overline{H}_n^{T-1} \tilde{v}_n,$$
$$\Delta \tilde{v}_n = \overline{H}_{n+1}^{T} C_n \overline{H}_{n+1} \tilde{u}_{n+1},$$

which is equivalent to the second order system

$$\Delta(\tilde{B}_n^{-1} \Delta \tilde{u}_n) - \tilde{C}_n \tilde{u}_{n+1} = 0, \tag{3.12}$$

where $\tilde{B}_n = \overline{H}_n^{-1} B_n \overline{H}_n^{T-1}$, $\tilde{C}_n = \overline{H}_{n+1}^{T} C_n \overline{H}_{n+1}$. Let $\tilde{H}_n$ be given recurrently by

$$\tilde{H}_M = I, \qquad \tilde{H}_{n+1} = \tilde{B}_n \tilde{H}_n^{T-1}, \qquad n \geq M.$$

Then $\tilde{H}_{n+1}^{T} \tilde{B}_n^{-1} \tilde{H}_n = I$ and the transformation $\tilde{u}_n = \tilde{H}_n u_n$ transforms (3.12) into the system

$$\Delta^2 u_n + \tilde{H}_{n+1}[\Delta(\tilde{B}_n^{-1} \Delta \tilde{H}_n) - \tilde{C}_n \tilde{H}_{n+1}] u_{n+1} = 0. \tag{3.13}$$

Indeed, we have

$$\begin{aligned}\tilde{H}^T_{n+1}[\Delta(\tilde{B}_n^{-1}\Delta\tilde{u}_n) - \tilde{C}_n\tilde{u}_n] &= \tilde{H}^T_{n+1}[\Delta(\tilde{B}_n^{-1}\tilde{H}_n\Delta u_n \\ &+ \tilde{B}_n^{-1}\Delta\tilde{H}_n u_{n+1}) - \tilde{C}_n\tilde{H}_n u_{n+1}] = \Delta(\tilde{H}^T_{n+1}\tilde{B}_n^{-1}\tilde{H}_n\Delta u_n) \\ &+ \tilde{H}^T_{n+1}[\Delta(\tilde{B}_n^{-1}\Delta\tilde{H}_n) - \tilde{C}_n\tilde{H}_{n+1}]u_{n+1} + (\tilde{H}^T_{n+1}\tilde{B}^{-1}_{n+1}\Delta\tilde{H}_{n+1} \\ &- \Delta H^T_{n+1}\tilde{B}^{-1}_{n+1}\tilde{H}_{n+1})\Delta u_{n+1} = \Delta^2 u_n + P_n u_{n+1},\end{aligned}$$

where $P_n = \tilde{H}_{n+1}[\Delta(\tilde{B}_n^{-1}\Delta\tilde{H}_n) - \tilde{C}_n\tilde{H}_{n+1}]$. Consequently, transformation (3.4) with $H_n = \bar{H}_n\tilde{H}_n$, $K_n = \bar{H}^{T-1}\tilde{B}_n^{-1}\Delta\tilde{H}_n$ transforms (1.1) into (3.13). ∎

4. Discrete Singular Quadratic Functionals

Consider the quadratic functional associated with (1.1) (see Proposition 2.1)

$$q(y_n, z_n, M, N) = \sum_{k=M}^{N}(z_n^T B_n z_n + y^T_{n+1}C_n y_{n+1}) \tag{4.1}$$

over $\{y_n, z_n\} \in \Omega(M, \infty)$, where

$$\Omega(M,\infty) = \{y_n, z_n \in R^d, y_M = 0 = \lim_{n\to\infty} y_n, \Delta y_n = A_n y_{n+1} + B_n z_n\}.$$

In this section we look for conditions under which

$$q(y_n, z_n; M, \infty) := \liminf_{N\to\infty} q(y_n, z_n; M, N) \geq 0 \tag{4.2}$$

for every $\{y_n, z_n\} \in \Omega(M, \infty)$ (such a pair of d-dimensional sequences we will call admissible for (4.2)).

First we study the influence of transformation (3.4) with $M_n \equiv 0$ on the quadratic functional (4.1).

THEOREM 4.1. *Let $\Delta y_n = A_n y_{n+1} + B_n z_n$, $y_n = H_n u_n$, $z_n = K_n u_n + H_n^{T-1}v_n$, where H_n, $H_n + B_n K_n$ are nonsingular and $H_n^T K_n = K_n^T H_n$, $n \in [M, N+1]$. Then*

$$q(y_n, z_n; M, N) = u_n^T H_n^T K_n u_n\big|_M^{N+1} + \sum_M^n (v_n^T\tilde{B}_n v_n + u^T_{n+1}\tilde{C}_n u_{n+1}),$$

where

$$\Delta u_n = \tilde{A} u_{n+1} + \tilde{B}_n v_n \tag{4.3}$$

and $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ *are given by* (3.9).

Proof. The validity of (4.2) follows from Corollary 3.1. Further,

$$\begin{aligned} q(y_n, z_n; M, N) &= \sum_{n=M}^{N} (z_n^T B_n z_n + y_{n+1}^T C_n y_{n+1}) = z_n^T y_n|_M^{N+1} \\ &\quad - \sum_M^N y_{n+1}^T(\Delta z_n - C_n y_{n+1} + A_n^T z_n) = u_n^T H_n^T(K_n u_n \\ &\quad + H_n^{T-1} v_n)|_M^{N+1} - \sum_M^N u_{n+1}^T[\Delta v_n - \tilde{C}_{n+1} u_{n+1} + \tilde{A}_n^T v_n] \\ &= u_n^T H_n^T K_n u_n|_M^{N+1} + u_n^T v_n|_M^{N+1} - u_n^T v_n|_M^{N+1} \\ &\quad + \sum_M^N (v_n^T \tilde{B}_n v_n + u_{n+1}^T \tilde{C}_n u_{n+1}) \\ &= u_n^T H_n K_n u_n|_M^{N+1} + \sum_M^N (v_n^T \tilde{B}_n v_n + u_{n+1}^T \tilde{C}_n u_{n+1}). \end{aligned}$$

Here we have used the equality

$$H_{n+1}^T(\Delta z_n - C_n y_{n+1} + A_n^T z_n) = \Delta v_n - \tilde{C}_n u_{n+1} + \tilde{A}_n^T v_n,$$

which follows from (4.3) and the computations given in the proof of Corollary 3.1. ∎

By a similar method as in [5] we may now prove the following theorem.

THEOREM 4.2. *Suppose that* (A), (B) *from Section* 2 *hold.*

(i) *If* (1.1) *is disconjugate on* $[M-1, \infty)$ *and*

$$\liminf_{n\to\infty} \; y_n^T Z_n Y_n^{-1} y_n \geq 0 \tag{4.4}$$

for every $\{y_n, z_n\} \in \Omega(M, \infty)$, *where* $\{Y_n, Z_n\}$ *is the solution of* (2.1) *satisfying* $Y_{M-1} = 0$, $Z_{M-1} = I$, *then* (4.1) *holds with equality only for* $\{y_n, z_n\} = \{0, 0\}$.

(ii) *Suppose that* (4.2) *holds for* $\{y_n, z_n\} \in \Omega(M, \infty)$. *Then* (1.1) *is disconjugate on* $[M, \infty)$ *and* (4.4) *holds for any* $\{y_n, z_n\} \in \Omega(M, \infty)$, *for which* $q(y_n, z_n; M, \infty) < \infty$.

Proof. (i) Let $\{Y_n, Z_n\}$ be the solution of (2.1) for which $Y_{M-1} = 0$, $Z_{M-1} = I$ and let $y_n = Y_n u_n$, $z_n = Z_n u_n + Y_n^{T-1} v_n$. Since (1.1) is disconjugate on $[M-1,\infty)$, we have $Y_n^T B_n^{-1}(I-A_n)Y_{n+1}>0$ for every $n\ge M$. By Theorem 4.1 and Corollary 3.2 for any $\{y_n, z_n\}\in\Omega(M,\infty)$

$$q(y_n, z_n; M, N) = y_n^T Z_n Y_n^{-1} y_n\Big|_M^{N+1} + \sum_M^N v_n^T Y_{n+1}^{-1}(I-A_n)^{-1} B_n Y_n^{T-1} v_n$$

$$= y_{N+1}^T Z_{N+1} Y_{N+1}^{-1} y_{N+1} + \sum_M^N v_n^T [Y_n^T B_n^{-1}(I-A_n)Y_{n+1}]^{-1} v_n.$$

Consequently, if (4.4) holds, then for $\{y_n, z_n\}\in\Omega(M,\infty)$

$$q(y_n, z_n; M, \infty) = \liminf_{N\to\infty} q(y_n, z_n, M, N)\ge 0$$

with equality only for $v_n\equiv 0$, i.e., $\{y_n, z_n\}\equiv\{0,0\}$.

(ii) Let $\{Y_n, Z_n\}$ be the solution of (2.1) for which $Y_M=0$, $Z_M=I$ and suppose that there exists $K>M$ such that $\eta^T Y_K^T B_K^{-1}(I-A_K)Y_{K+1}\eta\le 0$ for some $\eta\in\mathbb{R}^d$. If $\eta^T Y_K^T B_K^{-1}(I-A_K)Y_{K+1}\eta<0$, define

$$\{y_n, z_n\} = \begin{cases} \{Y_n\eta, Z_n\eta\}, & n\in[M, K-1],\\ \{Y_K\eta - B_K^{-1}Y_K\eta\}, & n=K,\\ \{0,0\}, & n\ge K+1.\end{cases}$$

Then y_n, z_n is admissible for (4.1) and

$$\begin{aligned} q(y_n, z_n; M,\infty) &= q(y_n, z_n; M, K-1) + z_K^T B_K z_K\\ &= \eta Y_k^T B_K^{-1} B_K B_K^{-1} Y_K\eta + \sum_{n=M}^{K-1}\eta^T(Z_n^T B_n Z_n + Y_{n+1}^T C_n Y_{n+1})\eta\\ &= \eta^T Y_K^T B_K^{-1} Y_K\eta + \sum_{n=M}^{K-1}\eta^T[\Delta(Y_n^T Z_n)\\ &\quad + (-\Delta Z_n + C_n Y_{n+1} - A_n^T Z_n)]\eta = \eta^T Y_K^T B_K^{-1} Y_K\eta\\ &\quad + \eta^T Y_K^T Z_K\eta = \eta^T Y_K^T B_K^{-1}(Y_K - Y_K + (I-A_K)Y_{K+1})\eta\\ &= \eta^T Y_K B_K^{-1}(I-A_K)Y_{K+1}\eta < 0,\end{aligned}$$

a contradiction (this construction of the test sequences $\{y_n, z_n\}$ is the same as in the proof of Proposition 2.1 in [8]). Now suppose that $\eta^T Y_K^T B_K^{-1}(I-A_K)Y_{K+1}\eta=0$. Then $Y_K\eta=0$ or $Y_{K+1}\eta=0$. Consider the first case (the second case is analogical) and define

$$\{y_n, z_n\} = \begin{cases} \{Y_n\eta, Z_n\eta\}, & M \le n \le K-1, \\ \{0, 0\}, & m \ge K. \end{cases}$$

Then $\{y_n, z_n\} \in \Omega(M, K-1)$ and $q(y_n, z_n; M, K) = 0$. Similarly as in the continuous case (see [14, p. 324]) the following holds: If $q(\eta_n, \xi_n; M, K) \ge 0$ for every $\{\eta_n, \xi_n\} \in \Omega(M, K-1)$ and $q(\tilde{y}_n, \tilde{z}_n; M, K) = 0$ for some $\{\tilde{y}_n, \tilde{z}_n\} \in \Omega(M, K-1)$, then $\{\tilde{y}_n, \tilde{z}_n\}$ is a solution of (1.1) on $[M, K]$. Since $q(y_n, z_n; M, \infty) \ge 0$ over $\Omega(M, \infty)$ we have $q(y_n, z_n; M, K) \ge 0$ over $\Omega(M, K-1)$, and hence $\{y_n, z_n\}$ is an extremal, i.e., a solution of (1.1). But this is a contradiction since the only extremal for which $\{y_n, z_n\} = \{0, 0\}$ is the trivial one. Consequently, $Y_K^T B_K^{-1}(I - A_K)Y_{K+1} > 0$ and by Proposition 2.1 system (1.1) is disconjugate on $[M, \infty)$.

Now suppose that there exists $\{\tilde{y}_n, \tilde{z}_n\} \in \Omega(M, \infty)$ such that $q(\tilde{y}_n, \tilde{z}_n; M, \infty) < \infty$ and

$$\liminf_{n\to\infty} \tilde{y}_n^T Z_n Y_n^{-1} \tilde{y}_n \le -\varepsilon < 0, \tag{4.5}$$

where $\{Y_n, Z_n\}$ is the solution of (2.1) satisfying $Y_M = 0$, $Z_M = I$. To get a contradiction we proceed similarly as in the continuous case (cf. [5, 15]). Define the test sequence $\{y_n, z_n\}$ as follows:

$$\{y_n, z_n\} = \begin{cases} \{Y_n Y_N^{-1}\tilde{y}_N, Z_n Y_N^{-1}\tilde{y}_N\}, & n \in [M, N-1], \\ \{\tilde{y}_n, \tilde{z}_n\}, & n \ge N. \end{cases}$$

Directly one may verify that $\{y_n, z_n\} \in \Omega(M, \infty)$ and for $K > N$ we have

$$\begin{aligned} q(y_n, z_n; M, K) &= q(y_n, z_n; M, N-1) + q(y_n, z_n; N, K) \\ &= \sum_{n=M}^{N-1} \tilde{y}_N Y_N^{T-1}[Z_n^T B_n Z_n + Y_{n+1}^T C_n Y_{n+1})\eta] Y_N^{-1}\tilde{y}_N \\ &\quad + \sum_{n=N}^{K} (\tilde{z}_n^T B_n \tilde{z}_n + \tilde{y}_{n+1}^T C_n \tilde{y}_{n+1}) \\ &= \tilde{y}_N Y_N^{T-1} \sum_{n=M}^{N-1} [\Delta(Y_n^T Z_n) + Y_{n+1}^T(-\Delta Z_n + C_n Y_{n+1} \\ &\quad - A_n^T Z_n)] Y_N^{-1}\tilde{y}_N = \sum_{n=N}^{K} (\tilde{z}_n^T B_n \tilde{z}_n + \tilde{y}_{n+1}^T C_n \tilde{y}_{n+1}) \\ &\quad + \tilde{y}_N^T Z_N Y_N^{-1}\tilde{y}_N + \sum_{n=N}^{K} (\tilde{z}_n B_n \tilde{z}_n + \tilde{y}_{n+1} C_n \tilde{y}_{n+1}). \end{aligned}$$

Now, (4.5) and $q(\tilde{y}_n, \tilde{z}_n; M, \infty) < \infty$ imply that N can be chosen such that $\tilde{y}_N^T Z_N Y_N^{-1} \tilde{y}_N < -2\varepsilon/3$ and $\liminf_{K\to\infty} \sum_{n=N}^{K} (\tilde{z}_n^T B_n \tilde{z}_n + \tilde{y}_{n+1}^T C_n \tilde{y}_{n+1}) < \varepsilon/3$. Consequently $q(y_n, z_n; M, \infty) < 0$, a contradiction. ▮

Remark 4.1. Similar to [5, 15] (continuous case) one may look for conditions on matrices A_n, B_n, C_n which guarantee the validity of singularity condition (4.3). This condition is satisfied, e.g., $\liminf_{n\to\infty} W_n > -\infty$ (this inequality means that $\liminf_{n\to\infty} \lambda(W_n) > \infty$ for any sequence of eigenvalues $\lambda(W_n)$ of W_n, where W_n is the so-called distinguished solution at ∞ of (2.3); for terminology see [2]). We hope to follow this idea elsewhere.

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